



DETERMINATION OF THE FREQUENCIES AND FORMS OF OSCILLATIONS OF NON-UNIFORM DISTRIBUTED SYSTEMS WITH BOUNDARY CONDITIONS OF THE THIRD KIND†

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An effective method of constructing a solution of the eigenvalue and eigenfunction problem for an essentially non-uniform oscillatory system with variable distributed parameters is developed. The proposed approach is based on a combination of a variational approach, the theory of boundary-value problems and perturbation methods. An original determination of the small parameter of the problem is given and a recurrence algorithm for the successive refinement of the eigenvalues and eigenfunctions is proposed, which leads to accelerated (quadratic) convergence of the Newton's "method of tangents" type. A commentary is made on the method and is illustrated by the solution of model examples. © 1997 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider the natural oscillations of a non-uniform system with distributed parameters (a string, a spindle or a beam). We will assume that the condition for elastic clamping is satisfied at its boundary (at the ends). Using the method of separation of variables, we arrive at the third eigenvalue and eigenfunction boundary-value problem of the form [1-4]

$$\begin{aligned} (p(x)u')' + [\lambda r(x) - q(x)]u &= 0, \quad 0 < x < 1 \\ \alpha_0 p(0)u'(0) - \beta_0 u(0) &= 0, \quad \alpha_0, \beta_0 \geq 0, \quad \alpha_0 + \beta_0 > 0 \\ \alpha_1 p(1)u'(1) + \beta_1 u(1) &= 0, \quad \alpha_1, \beta_1 \geq 0, \quad \alpha_1 + \beta_1 > 0 \end{aligned} \tag{1.1}$$

Here $u = u(x)$ is a coordinate function characterizing the form, in particular, the linear or angular displacement, of the section x , x is the argument (the Euler variable), $0 \leq x \leq 1$, the parameter λ is the constant of the separation of the spatial and time variables $\lambda = \omega^2 \geq 0$, and ω is the frequency. The coefficients $p(x)$ and $r(x)$ have the meaning of the distributed stiffness and linear density, respectively; it is assumed that they are bounded and strictly positive. The non-negative function $Q(x)$ is the elasticity coefficient of the external medium (in particular, when $q \equiv 0$ there is no medium). The functions p , r and q are assumed to be fairly smooth (see below).

Note that the Sturm-Liouville problem (1.1) with boundary conditions of the third kind is written in dimensionless variables. We take as the unit of length the extension of the system (the string, elastic spindle or beam, etc.), and we take as the units of the stiffness, density, elasticity coefficients, etc. their characteristic values (the average or extremal values).

For the self-conjugate boundary-value problem (1.1) it is required to determine the system of eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{u_n(x)\}$ ($n = 1, 2, \dots$). From the applied point of view, a highly accurate calculation of the lower frequencies $\omega_1, \omega_2, \dots$ and forms $u_1(x), u_2(x), \dots$ of the oscillations, which govern the system performance, it is extremely important ([1-5], etc.). To determine the higher frequencies $\omega_n, n \gg 1$ and the corresponding forms $u_n(x)$ we can use asymptotic methods [3, 4, 6]. The problem of finding the first natural frequencies and forms leads to fundamental analytic and computational difficulties in the case of systems that are essentially non-uniform, when the coefficients $p(x), r(x)$ and $q(x)$ vary considerably for $0 \leq x \leq 1$, see, for example, Section 5.2. Note that the form of Eq. (1.1) can be simplified using various substitutions [3, 7].

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An enormous literature is devoted to investigating the eigenvalue and eigenfunction problem (1.1) (see, in particular, the monographs and textbooks [1–10] and the bibliography in them). Numerous analytic and numerical methods of solving it have been developed. However, these investigations are mainly devoted to the case of boundary conditions of the first kind ($\alpha_0 = \alpha_1 = 0$). The case of conditions of the second kind ($\beta_0 = \beta_1 = 0$) have been much less investigated. The general situation of boundary conditions of the third kind has not been investigated very thoroughly. Using the variational approach, we intend to develop an effective numerical–analytic method of determining the lower frequencies and forms of oscillations of a system with essentially non-uniform (variable with respect to x) parameters p , r , and q . The proposed algorithm for solving the problem touches on the investigations made in [11, 12] in which the case of boundary conditions of the first and second kinds were considered.

2. THE VARIATIONAL APPROACH TO THE SOLUTION OF THE PROBLEM

Problem (1.1) can be represented in the equivalent variational form [3, 4, 9, 10]

$$\begin{aligned} \min J[u] = \lambda, \quad J[u] &= \int_0^1 [p(x)u'^2 + q(x)u^2] dx - p(x)u'u \Big|_0^1 \\ I[u] = \|u\|^2 &= \int_0^1 r(x)u^2 dx = 1, \quad [\alpha_{0,1}p(x)u' \mp \beta_{0,1}u]_{x=0,1} = 0 \end{aligned} \quad (2.1)$$

Relations (1.1) are the necessary and sufficient conditions for the functional $J[u]$ (2.1) to have a minimum for the isoperimetric normalization condition with weight $r(x)$: $I[u] = \|u\|^2 = 1$ and boundary conditions of the third kind. The absolute minimum of the functional is the first eigenvalue λ_1 , while the function $u(x, \lambda_1)$, on which it is reached, is the first eigenfunction $u_1(x)$ of the problem. Subsequent eigenvalues λ_n , $n \geq 2$ are determined recurrently in narrower classes of functions, which satisfy additional conditions of orthogonality with weight $r(x)$ to the previous eigenfunctions u_1, \dots, u_{n-1}

$$I_k[u] = \int_0^1 r(x)u_k(x)u dx = (u_k, u) = 0, \quad k = 1, \dots, n-1 \quad (2.2)$$

The solution of variational problem (2.1), (2.2) leads to a system of eigenvalues $\{\lambda_n\}$ and orthonormalized functions $\{u_n(x)\}$. The solution of the Sturm–Liouville problem (1.1) defines the eigenfunctions (apart from arbitrary factors, which can be chosen from the normalization condition with weight $r(x)$).

Suppose λ , $u(x, \lambda)$ is a certain solution of problem (1.1). Multiplying Eq. (1.1) by $u(x, \lambda)$ and integrating in the interval $0 \leq x \leq 1$, we obtain $\lambda = J[u]/I^{-1}[u]$ (taking the boundary conditions into account). This relation is used in the Rayleigh principle [3–5, 8–12] to construct an upper limit of the first eigenvalue λ_1 using a certain test function $\psi(x)$, which satisfies the boundary conditions (1.1)

$$0 < \lambda_1 \leq \lambda_1^* = J[\psi]/\|\psi\|^{-2}, \quad [\alpha_{0,1}p(x)\psi'(x) \mp \beta_{0,1}\psi(x)]_{x=0,1} = 0 \quad (2.3)$$

The accuracy of the limit λ_1^* (2.3) depends on the successful choice of the test function $\psi(x)$ and will usually be fairly high if the properties of the functions $p(x)$, $r(x)$ and $q(x)$ are taken into account intuitively (see the solution of the examples below in Section 5). The equality $\lambda_1^* = \lambda_1$ is obtained if and only if $\psi(x) = Cu_1(x)$, where $C = \text{const}$. Note that in the estimate (2.3) the property of the normalizability of the function $\psi(x)$ is unnecessary.

Using the Rayleigh–Ritz method [3–10], justified in [8], we can obtain as high an accuracy of the limit λ_1^* and the subsequent limits λ_2^* , λ_3^* , . . . as desired. However, this approach involves extremely time-consuming and unstable calculations. In specific calculations, to confirm the acceptable accuracy it is necessary to construct analogous lower limits λ_{n*} ($n = 1, 2, 3, \dots$). The construction of the lower limits involves considerable theoretical and computational difficulties [9, 10]. It is of considerable practical importance to find these limits.

3. THE USE OF THE PERTURBATION METHOD

To fix our ideas, we will now consider the problem of obtaining the first eigenvalue λ and eigenfunction $u(x)$. The number $n = 1$ is omitted for brevity; this does not lead to any misunderstandings. We will assume that there is an approximate value λ° of the first eigenvalue, in particular $\lambda^\circ = \lambda^*$ (the upper limit), that is sufficiently accurate for the perturbation method to be applicable. As a measure of closeness we will take the value of the parameter ε , defined for each λ° by the following procedure

$$\begin{aligned} \varepsilon &= 1 - \xi, \quad |\varepsilon| \ll 1, \quad \xi = \min_x \arg E(x, \lambda^\circ) > 0 \\ E(x, \lambda^\circ) &= E^\circ(x) = \alpha_1 p(x) v'(x, \lambda^\circ) + \beta_1 v(x, \lambda^\circ) \\ (p(x)v')' + [\lambda^\circ r(x) - q(x)]v &= 0, \quad 0 < x \leq \xi \\ 1) \quad v(0) &= \alpha_0 a, \quad v'(0) = \beta_0 a/p(0) \quad (a = 1) \\ 2) \quad v(0) &= \alpha_0 p(0)b, \quad v'(0) = \beta_0 b \quad (b = 1) \end{aligned} \quad (3.1)$$

where $\arg_x E$ denotes the root x of the equation $E = 0$.

According to (3.1) it is required to solve the Cauchy problem for one of the equivalent forms of conditions 1 and 2 for $x = 0$, which are appropriate both when $\alpha_0 = 0$ ($\beta_0 > 0$) and when $\beta_0 = 0$ ($\alpha_0 > 0$). In case 1 $v(0)$ is specified, and $v'(0)$ is calculated using (1.1) and (3.1); in case 2, the opposite is the case; the function v will contain parameter a or parameter b , respectively, as a factor, which can be equated to unity. The quantity $E^\circ(x)$ represents the boundary condition in the region of the right end $x = 1$. Note that $E^\circ(1) = 0$ only when $\lambda^\circ = \lambda$, and $\varepsilon \rightarrow 0$ if $\lambda^\circ \rightarrow \lambda$. Moreover, we have the estimates $|\lambda - \lambda^\circ| \leq D|\varepsilon|$, $|\varepsilon| \leq d|\lambda - \lambda^\circ|$, where $0 < D, d < \infty$. This property is used for the assumed measure of closeness of λ and λ° .

In relations (3.1) we have assumed the continuability of the functions $p(x)$, $r(x)$ and $q(x)$ in the interval $1 \leq x \leq \xi$ when $\xi > 1$. Further, this continuation is assumed to be fairly smooth. Note that when $q(x) \equiv 0$ (there is no external elastic medium) the quantity $\xi < 1$ if $\lambda^\circ = \lambda^* > \lambda$ [3, 7, 11, 12]. The solution of problem (3.1) will henceforth be assumed known: $v = v(x, \lambda^\circ)$, $\xi = \xi(\lambda^\circ)$.

The smallness of the value of the parameter ε , defined by (3.1), can be achieved using the Rayleigh–Ritz method [8]. Computational practice shows that, to satisfy the conditions of perturbation theory it is usually sufficient to have $|\varepsilon| \sim 10^{-1} - 10^{-2}$ (see Section 5). As a rule, this estimate is obtained using Rayleigh's principle (2.3).

Thus, suppose the parameter ε is sufficiently small; we carry out identical transformations of the boundary-value problem (1.1) by replacing the argument, function and parameter

$$\begin{aligned} y &= x\xi, \quad 0 \leq y \leq \xi, \quad U(y, \varepsilon) \equiv u(x), \quad \Lambda = \xi^{-2}\lambda \\ (p(y)(1-\varepsilon)^{-1}U')' + [\Lambda r(y(1-\varepsilon)^{-1}) - (1-\varepsilon)^{-2}q(y(1-\varepsilon)^{-1})]U &= 0 \\ \alpha_0 p(0)(1-\varepsilon)U'(0, \varepsilon) - \beta_0 U(0, \varepsilon) &= 0 \\ \alpha_1 p(\xi + \varepsilon)(1-\varepsilon)U'(\xi, \varepsilon) + \beta_1 U(\xi, \varepsilon) &= 0 \end{aligned} \quad (3.2)$$

Relations (3.2) are considered as a perturbed problem in the eigenvalues Λ and eigenfunctions U for values of y in the range $0 \leq y \leq \xi$. We will take as the unperturbed (generating) problem the equation and boundary conditions for $\varepsilon = 0$

$$\begin{aligned} (p(y)U'_0)' + [\Lambda_0 r(y) - q(y)]U_0 &= 0, \quad 0 \leq y \leq \xi \\ \alpha_0 p(0)U'_0(0) - \beta_0 U_0(0) &= 0 \\ \alpha_1 p(\xi)U'_0(\xi) + \beta_1 U_0(\xi) &= 0 \end{aligned} \quad (3.3)$$

i.e. boundary-value problem (1.1) in the interval $0 \leq y \leq \xi$.

The solution of the Sturm–Liouville problem (3.3) Λ_0, U_0 is known; it is identical with the solution of the Cauchy problem constructed using (3.1), namely

$$\Lambda_0 = \lambda^\circ, \quad U_0(y) = v(y, \lambda^\circ), \quad 0 \leq y \leq \xi \quad (3.4)$$

For the initial perturbed problem (3.2), we will construct the required solution in the form of the representation

$$\Lambda(\varepsilon) = \lambda^\circ + \varepsilon\Lambda_1 + \varepsilon^2 \dots, \quad U(y, \varepsilon) = v(y, \lambda^\circ) + \varepsilon U_1 + \varepsilon^2 \dots \quad (3.5)$$

The unknowns Λ_1, U_1 satisfy the inhomogeneous boundary-value problem, which is obtained after substituting (3.5) into (3.2) and equating coefficients of the first power of ε

$$\begin{aligned} & (p(y)U_1')' + [\lambda^\circ r(y) - q(y)]U_1 = -\Lambda_1 r(y)U_0(y) - \\ & -(yp'(y)U_0'(y))' - \lambda^\circ yr'(y)U_0(y) + 2q(y)U_0(y) + yq'(y)U_0(y) \\ & \alpha_0 p(0)U_1'(0) - \beta_0 U_1(0) = \beta_0 U_0(0) \\ & \alpha_1 p(\xi)U_1'(\xi) + \beta_1 U_1(\xi) = -\alpha_1 p'(\xi)U_0'(\xi) - \beta_1 U_0(\xi) \end{aligned} \quad (3.6)$$

To determine the unknowns $\Lambda_1, U_1(y)$ we will use the property of self-conjugacy of the corresponding homogeneous problem (the Fredholm alternative). We multiply the equation for U_1 (3.6) by U_0 and integrate with respect to y in the interval $0 \leq y \leq \xi$. As a result of integration by parts, terms containing U_1 vanish. The coefficient of U_1 on the left-hand side of the relation is identically equal to zero, when boundary conditions (3.6) are taken into account. A linear equation for Λ_1 is obtained, by solving which we find the required value in terms of known quantities

$$\begin{aligned} \Lambda_1 = & \|U_0\|^{-2} \int_0^\xi yp'(y)U_0'^2(y)dy + \|U_0\|^{-2} \int_0^\xi [-\lambda^\circ yr'(y) + \\ & + 2q(y) + yq'(y)]U_0^2(y)dy + \|U_0\|^{-2} [(\beta_1 / \alpha_1)U_0^2(\xi) + (\beta_0 / \alpha_0)U_0^2(0)] \end{aligned} \quad (3.7)$$

When calculating Λ, U to a first approximation in ε (with an error of $O(\varepsilon^2)$) we can assume that $\xi = 1$ in (3.7) without any reduction in accuracy with respect to powers of ε . Hence, the continuability of the functions p, r and q may only be required to solve problem (3.1).

Following the definition of Λ (3.2) and its representation in the form of expansion (3.5), we obtain $\lambda^\circ = \lambda^*$, so that the quantity λ^* will be a lower limit for λ when a certain inequality in $\varepsilon\Lambda_1$ is satisfied, i.e.

$$0 < \lambda_* = \xi^2 \lambda^* < \lambda, \quad \varepsilon\Lambda_1 > 0 \quad (3.8)$$

The quantity $|\varepsilon|$ is assumed to be fairly small. It follows from (3.8) that the simultaneous satisfaction of the inequalities $\xi(\lambda^*) > 1$ (i.e. $\varepsilon < 0$) and $\varepsilon\Lambda_1 > 0$ (i.e. $\Lambda_1 < 0$) is impossible. The sign of the number ε is defined by (3.1), and the sign of the coefficient Λ_1 is found using (3.7). In particular, it can be established without integration that $\Lambda_1 > 0$ when $p'(y) \geq 0, r'(y) \leq 0$ and $q'(y) \geq 0, 0 \leq y \leq 1$. It should be noted that the terms outside the integrals in (3.7) are equal to zero when $\alpha_{0,1} = 0$ or $\beta_{0,1} = 0$.

Thus, suppose the coefficient Λ_1 is calculated from (3.7); the function $U_1(y)$ can then be constructed as the solution of problem (3.6) using the method of variation of the integration constants. For this purpose a general solution of the homogeneous equation is obtained which is found by means of Liouville's formula [7]. Then, in the same way as above, the subsequent coefficients Λ_2, U_2, \dots of expansions (3.5) are found. One can also use the method of successive approximations in powers of ε of the Picard-method type. However, this approach is extremely time consuming and unproductive. Below we develop an effective economic method of accelerated convergence, which leads to simple uniform computer calculations: integration of the Cauchy problem and determination of the root. Note also that in practical situations it is often sufficient to calculate the first approximation using (3.5) and (3.7) when constructing bilateral estimates of the type (2.3) and (3.8). The main results obtained and the assertions made also hold for the subsequent eigenvalues, since the specific features of the first eigenvalue are not used.

4. THE METHOD OF ACCELERATED CONVERGENCE

The cumbersome expression for Λ_1 (3.7), which contains quadratures of functions usually determined numerically as a result of integrating the Cauchy problem (3.1), can be simplified considerably. We will integrate them by parts, dispensing with the derivatives p' , r' and q' . After fairly lengthy elementary calculations, taking Eqs (3.1) and (3.3) into account for $U_0 = v$, we obtain the following final expression for λ with an error of $O(\varepsilon^2)$

$$\lambda = \lambda^\circ + \varepsilon\mu + \varepsilon^2 \dots, \quad \mu = \mu(\lambda^\circ) = -\xi p(\xi) V_0'^2(\xi) - [\lambda^\circ r(\xi) - q(\xi)] V_0^2(\xi) \quad (4.1)$$

$$V_0(x) \equiv U_0(x) \|U_0\|^{-1}$$

Here $V_0(x)$ is the function $U_0(x) = v(x, \lambda_0)$, normalized with weight $r(x)$ in the interval $0 \leq x \leq \xi$. Again we note that without loss of accuracy with respect to powers of ε (in the first approximation in ε) the quantity ξ in μ can be put equal to unity ($\xi = 1$), i.e. continuation of the functions p , r and q and of the solution v and v' is not required (when $\xi > 1$). The norm of the function $U_0(x)$ is also related to the taking of a quadrature of the function $v(x, \lambda^\circ)$, determined by numerical integration in accordance with (3.1). This operation can be replaced by the procedure of simultaneous integration of the Cauchy problem for the variable v and its derivative $h = \partial v / \partial \lambda$

$$(p(x)h') + [\lambda^\circ r(x) - q(x)]h = -r(x)v, \quad h(0) = h'(0) = 0 \quad (4.2)$$

$$\|U_0\|^2 = \|v\|^2 = [v'(\xi, \lambda^\circ)h(\xi, \lambda^\circ) - v(\xi, \lambda^\circ)h'(\xi, \lambda^\circ)]p(\xi)$$

Formula (4.1) for μ can be considerably simplified if the boundary conditions are of the first or second kind [11, 12]. Then only one term remains (the first or the second). Hence, we have obtained a compact expression that is simple from the computational point of view for the refined eigenvalue λ (with an error $O(\varepsilon^2)$): $|\lambda - \lambda^{(1)}| \leq C\varepsilon^2$, where $\lambda^{(1)} = \lambda^\circ + \varepsilon\mu$, while the constant C in the range of values of ε can be effectively estimated in terms of the coefficients p , r and q .

We will again use relations (3.1) and (4.1) (and (4.2)) to construct a more accurate value of λ based on the $\lambda^{(1)}$ obtained and considered as the initial approximation (similar to λ^0). We substitute $\lambda^{(1)}$ into Eq. (3.1) instead of λ^0 , integrate the Cauchy problem (with conditions 1 or 2) $v_{(1)} = v(x, \lambda^{(1)})$ and determine the root $\xi_{(1)}$ of the equation $E(x, \lambda^{(1)}) = 0$, closest to the value $x = 1$. As a consequence of the mentioned simplicity of the root, for the parameter $\varepsilon_{(1)}$ and the solutions $v_{(1)}$, $v'_{(1)}$ we have the estimates

$$|\varepsilon_{(1)}| \leq d|\lambda - \lambda^{(1)}| \leq dC\varepsilon^2, \quad |v - v_{(1)}| \leq K|\varepsilon_{(1)}|, \quad |v' - v'_{(1)}| \leq K|\varepsilon_{(1)}| \quad (4.3)$$

$$0 < d, \quad C, \quad K < \infty, \quad 0 \leq x \leq \xi_{(1)}, \quad \xi_{(1)} = 1 - \varepsilon_{(1)}$$

We will use (4.1) and (4.2) to obtain the refined value of λ

$$\lambda^{(2)} = \lambda^{(1)} + \varepsilon_{(1)}\mu(\lambda^{(1)}), \quad |\lambda - \lambda^{(2)}| \leq C\varepsilon_{(1)}^2 \leq d^2C^3\varepsilon^4 \quad (4.4)$$

The process of refining ξ , ε , v , v' and λ is then repeated. The following recurrence relations are obtained

$$\lambda^{(k+1)} = \lambda^{(k)} + \varepsilon_{(k)}\mu(\lambda^{(k)}), \quad |\lambda - \lambda^{(k+1)}| \leq C\varepsilon_{(k)}^2$$

$$\varepsilon_{(k)} = 1 - \xi_{(k)}, \quad \xi_{(k)} = \min_x \arg E(x, \lambda^{(k)}) > 0$$

$$E(x, \lambda^{(k)}) = \alpha_1 p(x)v'(x, \lambda^{(k)}) + \beta v(x, \lambda^{(k)}) \quad (4.5)$$

$$|\varepsilon_{(k)}| \leq d|\lambda - \lambda^{(k)}| \leq dC\varepsilon_{(k-1)}^2, \quad |v - v_{(k)}| \leq K|\varepsilon_{(k)}|, \quad |v' - v'_{(k)}| \leq K|\varepsilon_{(k)}|$$

It follows from (4.5) (see, in particular, (4.3) and (4.4)), that the rate of convergence of the iterations is extremely high. It has a quadratic order in ε (similar to Newton's method of tangents)

$$|\varepsilon_{(k)}| \leq (dC)^{-1}(\varepsilon dC)^{\theta(k)}, \quad \theta(k) = 2^k, \quad k = 0, 1, 2, \dots \quad (4.6)$$

It follows from (4.6) that a few iterations (usually two or three) lead to highly accurate estimates of λ and u . The recurrent refinement procedure (4.5) can be employed to obtain successive eigenvalues λ_n and eigenfunctions $u_n(x)$, $n \geq 2$. It is attractive from the point of view of simplicity of realization, economy and high accuracy, and can easily be realized using modern personal computers.

We will consider once again the informal procedure of choosing the test function $\psi(x)$ or the system $\{\psi_i(x)\}$ and constructing the initial estimate λ^0 . If the quantity $|\varepsilon|$, more accurately $|\varepsilon|dC$, is "not very small", then, at the preliminary stage, one can use the well-developed "shooting" methods or the small-coordinate approximation by the Rayleigh–Ritz method. In practice, the functions p , r and q also often depend on the scalar or vector parameter γ , $\gamma \in \Gamma$ and it is required to construct families of eigenvalues and eigenfunctions. Suppose, for a certain value of $\gamma = \gamma^0 \in \Gamma$ a highly accurate (in particular, an exact) solution of the Sturm–Liouville problem is known: $\lambda = \lambda(\gamma^0)$, $u = u(x, \gamma^0)$. Then, these quantities can be taken as the initial approximations for other fairly close discrete values of $\gamma_j \in \{\gamma_j\}$, where $\{\gamma_j\}$ is a fairly dense set $\{\gamma_j\} \in \Gamma$. This method of continuation with respect to the parameter can also be realized by the artificial introduction of the parameter γ , $0 \leq \gamma \leq 1$, i.e. by replacing $p(x)$ by $P(x, \gamma)$, $r(x)$ by $R(x, \gamma)$ and $q(x)$ by $Q(x, \gamma)$, where $P(x, 0)$, $R(x, 0)$, $Q(x, 0)$ are "simple functions" (for example, constants), while $P(x, 1) \equiv p(x)$, $R(x, 1) \equiv r(x)$, $Q(x, 1) \equiv q(x)$. In particular, we can put

$$\begin{aligned} P(x, \gamma) &= p_0 + \gamma(p(x) - p_0), & R(x, \gamma) &= r_0 + \gamma(r(x) - r_0) \\ Q(x, \gamma) &= q_0 + \gamma(q(x) - q_0), & p_0, r_0 &> 0, & q_0 &\geq 0 \end{aligned}$$

where p_0 , r_0 , q_0 are certain characteristic values of the functions $p(x)$, $r(x)$ and $q(x)$ for $0 \leq x \leq 1$, for example, their average or extremal values. The parameters γ can take discrete values $\gamma_j = j/N$ ($j = 0, 1, 2, \dots, N$), where N must be taken to be fairly large, as established by numerical experiment.

5. CALCULATION OF MODEL EXAMPLES

In order to illustrate the effectiveness of the above method and to comment on the formulae and analytic expressions obtained, we will consider some examples.

5.1. *A system with constant coefficients.* In this special case, by dividing by $p > 0$ and redesignating the other parameters, problem (1.1) can be converted to the form

$$u'' + \lambda u = 0, \quad \alpha_0 u'(0) - \beta_0 u(0) = 0, \quad \alpha_1 u'(1) + \beta_1 u(1) = 0 \quad (5.1)$$

Suppose the coefficient β_0 or (and) β_1 is not equal to zero; to fix our ideas we will take $\beta_0 > 0$. Then, the eigenvalues λ_n and the functions $u_n(x)$, $n \geq 1$ of problem (5.1) are given by the relations

$$\begin{aligned} u_n(x) &= a(\sin v_n x + (\alpha_0 / \beta_0) \cos v_n x), \quad v_n = \lambda_n^{1/2}, \quad a = \text{const} \\ \lambda_n &= \text{Arg}_\lambda [(\beta_0 \beta_1 - \lambda \alpha_0 \alpha_1) \sin \lambda^{1/2} + (\alpha_0 \beta_1 + \beta_0 \alpha_1) \lambda^{1/2} \cos \lambda^{1/2}] \end{aligned} \quad (5.2)$$

where Arg_λ denotes the set of positive roots of λ .

The quantities λ_n can be found numerically. The characteristic equation for determining the positive eigenvalues λ_n (5.2) can be reduced to a form which is more usual and convenient for analysis

$$\text{tg } v = v(\alpha_0 \beta_1 + \beta_0 \alpha_1)(v^2 \alpha_0 \alpha_1 - \beta_0 \beta_1)^{-1}, \quad v^2 = \lambda \quad (5.3)$$

It can be established by direct differentiation with respect to v of relation (5.3) that the roots v_n are simple for all values of the parameters $\alpha_{0,1}$, $\beta_{0,1}$, which satisfy conditions of the type (1.1).

We will now calculate the implicit derivative $\partial \lambda_n / \partial \xi$, starting from (5.3). To avoid cumbersome calculations we will consider the simpler case when $\alpha_0 = 0$, i.e. the clamping of the left end is absolutely rigid, while the clamping of the right end is elastic. We obtain the expression

$$\partial \lambda_n / \partial \xi = -2(\alpha_1^2 \lambda_n^2 + \beta_1^2 \lambda_n)(\alpha_1 + \beta_1 \xi) \beta_1 + \alpha_1^2 \xi \lambda_n^{-1} < 0 \quad (5.4)$$

We will now use (4.1), according to which this derivative is determined by the coefficient μ , i.e.

$$\partial \lambda_n / \partial \xi = \mu(\lambda_n(\xi)) = -V_0'^2(\xi) - \lambda_n V_0^2(\xi) \quad (5.5)$$

Direct calculation of the value of the function $U_{n0}(x) = v(x, \lambda_n) = \sin v_n x$, its derivative and the square of the

norm $\|U_{n0}\|^2$ and substitution into the expression for μ shows that the values of (5.4) and (5.5) are identical. This provides a check that the main formula (4.1) holds in the special case of a homogeneous system with an elastically clamped right end. Note that for boundary conditions of the first or second kind (at the left or/and right ends) the value of the derivative is the same and equal to $\partial\lambda_n/\partial\xi = -2\lambda_n/\xi$.

5.2. *A calculation for an essentially inhomogeneous system.* We will consider the specific numerical example of determining the first eigenvalue for a problem of the form

$$u'' + \lambda(1 - 0.9 \sin \pi x)^{-1} u = 0, \quad u(0) = 0, \quad u'(1) + 2u(1) = 0 \quad (5.6)$$

Here the coefficient $r(x)$, see (1.1), is changed by a factor of 10, i.e. we are investigating a system with strongly varying parameters (the linear density). The equivalent variational problem has the form (2.1)

$$J[u] = \int_0^1 u'^2 dx + 2u^2(1) \rightarrow \min \quad (5.7)$$

$$I[u] = \int_0^1 (1 - 0.9 \sin \pi x)^{-1} u^2 dx = 1$$

with boundary conditions (5.6) on u . For the initial estimate of λ we will use the Rayleigh principle and we will take $\psi = \sin \pi x$ as the test function $\psi(x)$, where η is the first root of the equation $\eta + 2 \operatorname{tg} \eta = 0$, i.e. $\psi(x)$ will be sought in a form corresponding to the constant function $r(x)$. The boundary condition of the first kind at the left end is satisfied automatically, while the boundary condition of the third kind at the right end will be satisfied when $\eta = 2.28893$ (to five significant decimal places). A calculation of the upper limit using (2.3) leads to a value $\lambda^* = 1.12451$. Further, by integrating the Cauchy problem according to (3.1) with $v(0) = 0$, $v'(0) = 1$, we obtain the required value of the root $\xi = 0.93321$ of the function $E(x, \lambda^*) = v'(x, \lambda^*) + 2v(x, \lambda^*)$. As a result we obtain a value of the parameter $\varepsilon = 6.679 \times 10^{-2}$, which is sufficiently small for the perturbation method, and the method of accelerated convergence to be applicable. The refined eigenvalue given by (4.1) again leads to the upper limit $\lambda^{(1)} = 1.07907$ and to the corresponding root $\xi_{(1)} = 0.99170$ and the parameter $\varepsilon_{(1)} = 8.30 \times 10^{-3}$. Hence, by the first inequality of (4.3) we have $dC \approx 1.8$, i.e. quadratic convergence of process (4.5) of the type (4.6) with parameter $\varepsilon dC \approx 0.12$. The following iterations lead to values of the eigenvalue $\lambda^{(2)} = 1.0739955$, the root $\xi_{(2)} = 0.9991044$ and $\varepsilon_{(2)} = 8.956 \times 10^{-4}$ (to seven decimal places). Further calculations enable us to obtain $\lambda^{(3)} = 1.0734555$ and $\xi_{(3)} = 0.9999726$, i.e. $\varepsilon_{(3)} = 2.74 \times 10^{-5}$ and finally $\lambda_{(4)} = 1.0734400$.

A check shows (see Section 3 and formula (3.8)), that the quantity $\xi_{(3)}^2 \lambda^{(3)}$ is a lower limit; hence, the bilateral estimate $\lambda_* = \xi_{(3)}^2 \lambda^{(3)} = 1.073380 < \lambda < 1.073440$ holds.

We will take as the highly accurate approximate value of λ the arithmetic mean $1/2(\lambda_* + \lambda^{(4)})$, which leads to a relative error of $\Delta\lambda/\lambda \approx 2 \times 10^{-5}$. Note that the reduction of the accuracy of the calculations compared with the theoretical value is due to rounding errors.

We will now use the so-called two-coordinate approximation $\psi = c_1 \sin \eta_1 x + c_2 \sin \eta_2 x$ for the test function $\psi(x)$ where $\eta_1 = 2.28893$ and $\eta_2 = 5.08699$ are the first and second roots of the equation $\eta + 2 \operatorname{tg} \eta = 0$, respectively. As in Section 2 we will calculate the integrals $J[\psi]$ and $I[\psi]$ (5.7); we obtain the problem of minimizing the quadratic form $J(c)$ with the condition $I(c) = 1$, where $c = (c_1, c_2)^T$, while the functions $J(c)$ and $I(c)$ have the form

$$J(c) = c^T A c, \quad A = \operatorname{diag}(a_{11}, a_{22}), \quad a_{11} = 3.18666, \quad a_{22} = 13.80483$$

$$I(c) = c^T B c, \quad B = (b_{ij}), \quad b_{ij} = b_{ji}, \quad i, j = 1, 2 \quad (5.8)$$

$$b_{11} = 2.83381, \quad b_{12} = b_{21} = 0.91900, \quad b_{22} = 1.81257$$

Using the method of Lagrange multipliers, we arrive at the solutions of secular quadratic equation in λ : $\det(A - \lambda B) = 0$. The least root of the equation $\lambda^* = 1.09432$ is the upper limit of the first eigenvalue of problem (5.6). It is easy to see that it is more accurate than in the single-coordinate approximation of the test function $\psi(x)$ considered earlier. The determination of the abscissa ξ and the parameter ε using procedure (3.1) leads to the following values: $\xi = 0.97028$, $\varepsilon = 2.972 \times 10^{-2}$. As might have been expected, the two-coordinate approximation gives a more accurate result with respect to ε (by approximately a factor of two). We obtain as the first approximation of perturbation method (4.1) and (4.3) $\lambda^{(1)} = 1.0754351$, $\varepsilon_{(1)} = 1 - \xi_{(1)} = 3.0094 \times 10^{-3}$. Further, in a recurrent manner using (4.4) and (4.5) we obtain the following approximations: $\lambda^{(2)} = 1.0736612$, $\varepsilon_{(2)} = 1 - \xi_{(2)} = 3.284 \times 10^{-4}$, and, finally, $\lambda^{(3)} = 1.0734133$. The third iteration led to practically the same accuracy as the fourth for the single-coordinate approximation of $\psi(x)$. The analogous three-coordinate representation of the test function leads to the same relative error $\Delta\lambda/\lambda \approx 2 \times 10^{-5}$ as the second iteration. We again note that the reduction of the accuracy of the calculations compared with the theoretical value is due to the limitations of the microcalculator.

The results obtained above confirm that the proposed method of accelerated convergence is highly effective for the highly accurate solution of extremely complex eigenvalue and eigenfunction problems, without the need for expensive software.

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